1. Let $\Gamma$ be a circle with centre $O$. Let $\Lambda$ be another circle passing through $O$ and intersecting $\Gamma$ at points $A$ and $B$. A diameter $CD$ of $\Gamma$ intersects $\Lambda$ at a point $P$ different from $O$. Prove that
$$\angle APC = \angle BPD\,.$$

   **Solution.** Suppose that $A'$ is a point on $\Lambda$ such that $\angle A'PC = \angle BPD$. Then the segments $OA'$ and $OB$ subtends same angle in the respective minor arcs, so $OA' = OB$. This shows that $A$ lies on $\Gamma$ and hence $A' = A$. This proves that $\angle APC = \angle BPD$. □

2. Determine the smallest prime that does not divide any five-digit number whose digits are in a strictly increasing order.

   **Solution.** Note that 12346 is even, 3 and 5 divide 12345, and 7 divides 12348. Consider a 5 digit number $n = abcde$ with $0 < a < b < c < d < e < 10$. Let $S = (a+c+e)-(b+d)$. Then $S = a+(c-b)+(e-d) > a > 0$ and $S = e-(d-c)-(b-a) < e \leq 10$, so $S$ is not divisible by 11 and hence $n$ is not divisible by 11. Thus 11 is the smallest prime that does not divide any five-digit number whose digits are in a strictly increasing order. □

3. Given real numbers $a, b, c, d, e > 1$ prove that
$$\frac{a^2}{c-1} + \frac{b^2}{d-1} + \frac{c^2}{e-1} + \frac{d^2}{a-1} + \frac{e^2}{b-1} \geq 20\,.$$

   **Solution.** Note that $(a-2)^2 \geq 0$ and hence $a^2 \geq 4(a-1)$. Since $a > 1$ we have $\dfrac{a^2}{a-1} \geq 4$. By applying AM-GM inequality we get
$$\frac{a^2}{c-1} + \frac{b^2}{d-1} + \frac{c^2}{e-1} + \frac{d^2}{a-1} + \frac{e^2}{b-1} \geq 5\sqrt[5]{\frac{a^2b^2c^2d^2e^2}{(a-1)(b-1)(c-1)(d-1)(e-1)}} \geq 20\,.$$
□

4. Let $x$ be a non-zero real number such that $x^4 + \dfrac{1}{x^4}$ and $x^5 + \dfrac{1}{x^5}$ are both rational numbers. Prove that $x + \dfrac{1}{x}$ is a rational number.

   **Solution.** For a natural number $k$ let $T_k = x^k + 1/x^k$. Note that $T_4T_2 = T_2 + T_6$ and $T_8T_2 = T_{10} + T_6$. Therefore $T_2(T_8 - T_4 + 1) = T_{10}$. Since $T_{2k} = T_k^2 + 2$ it follows that $T_8, T_{10}$ are rational numbers and hence $T_2, T_6$ are also rational numbers. Since $T_5T_1 = T_4 + T_6$ it follows that $T_1$ is a rational number. □

5. In a triangle $ABC$, let $H$ denote its orthocentre. Let $P$ be the reflection of $A$ with respect to $BC$. The circumcircle of triangle $ABP$ intersects the line $BH$ again at $Q$, and the circumcircle of triangle $ACP$ intersects the line $CH$ again at $R$. Prove that $H$ is the incentre of triangle $PQR$.

**Solution.** Since $RACP$ is a cyclic quadrilateral it follows that $\angle RPA = \angle RCA = 90^\circ - \angle A$. Similarly, from cyclic quadrilateral $BAQP$ we get $\angle QPA = 90^\circ - \angle A$. This shows that $PH$ is the angular bisector of $\angle RPQ$.

We next show that $R, A, Q$ are collinear. For this, note that $\angle BPC = \angle A$. Since $\angle BHC = 180^\circ - \angle A$ it follows that $BHCP$ is a cyclic quadrilateral. Therefore $\angle RAP + \angle QAP = \angle RCP + \angle QBP = 180^\circ$. This proves that $R, A, Q$ are collinear.

Now $\angle QRC = \angle ARC = \angle APC = \angle PAC = \angle PRC$. This proves that $RC$ is the angular bisector of $\angle PRQ$ and hence $H$ is the incenter of triangle $PQR$. □

6. Suppose that the vertices of a regular polygon of 20 sides are coloured with three colours – red, blue and green – such that there are exactly three red vertices. Prove that there are three vertices $A, B, C$ of the polygon having the same colour such that triangle $ABC$ is isosceles.

   **Solution.** Since there are exactly three vertices, among the remaining 17 vertices there are nine of them of the same colour, say blue. We can divide the vertices of the regular 20-gon into four disjoint sets such that each set consists of vertices that form a regular pentagon. Since there are nine blue points, at least one of these sets will have three blue points. Since any three points on a pentagon form an isosceles triangle, the statement follows. □