1. Prove that there do not exist natural numbers x and y, with $x > 1$, such that

$$\frac{x^7 - 1}{x - 1} = y^5 + 1\,.$$

Solution. Simple factorisation gives $y^5 = x(x^3 + 1)(x^2 + x + 1)$. The three factors on the right are mutually coprime and hence they all have to be fifth powers. In particular, $x = r^5$ for some integer r. This implies $x^3 + 1 = r^{15} + 1$, which is not a fifth power unless $r = -1$ or $r = 0$. This implies there are no solutions to the given equation. □

2. In a triangle ABC, AD is the altitude from A, and H is the orthocentre. Let K be the centre of the circle passing through D and tangent to BH at H. Prove that the line DK bisects AC.

Solution. Note that $\angle KHB = 90°$. Therefore $\angle KDA = \angle KHD = 90° - \angle BHD = \angle HBD = \angle HAC$. On the other hand, if M is the midpoint of AC then it is the circumcenter of triangle ADC and therefore $\angle MDA = \angle MAD$. This proves that D, K, M are collinear and hence DK bisects AC. □

3. Consider the expression

$$2013^2 + 2014^2 + 2015^2 + \cdots + n^2\,.$$

Prove that there exists a natural number $n > 2013$ for which one can change a suitable number of plus signs to minus signs in the above expression to make the resulting expression equal 9999.

Solution. For any integer k we have

$$-k^2 + (k+1)^2 + (k+2)^2 - (k+3)^2 = -4\,.$$

Note that $9999 - (2013^2 + 2014^2 + 2015^2 + 2016^2 + 2017^2) = -4m$ for some positive integer m. Therefore, it follows that

$$\begin{aligned} 9999 =& (2013^2 + 2014^2 + 2015^2 + 2016^2 + 2017^2) \\ &+ \sum_{r=1}^{m} \left(-(4r + 2014)^2 + (4r + 2015)^2 + (4r + 2016)^2 - (4r + 2017)^2\right)\,. \end{aligned}$$

□

4. Let ABC be a triangle with $\angle A = 90°$ and $AB = AC$. Let D and E be points on the segment BC such that $BD : DE : EC = 1 : 2 : \sqrt{3}$. Prove that $\angle DAE = 45°$.

Solution. Rotating the configuraiton about A by $90°$, the point B goes to the point C. Let P denote the image of the point D under this rotation. Then $CP = BD$ and $\angle ACP = \angle ABC = 45°$, so ECP is a right-angled triangle with $CE : CP = \sqrt{3} : 1$. Hence $PE = ED$. It follows that $ADEP$ is a kite with $AP = AD$ and $PE = ED$. Therefore AE is the angular bisector of $\angle PAD$. This implies that $\angle DAE = \angle PAD/2 = 45°$. □

5. Let $n \geq 3$ be a natural number and let P be a polygon with n sides. Let $a_1, a_2, \ldots, a_n$ be the lengths of the sides of P and let p be its perimeter. Prove that

$$\frac{a_1}{p-a_1} + \frac{a_2}{p-a_2} + \cdots + \frac{a_n}{p-a_n} < 2\,.$$

Solution. If r and s are positive real numbers such that $r < s$ then $r/s < (r+x)/(s+x)$ for any positive real x. Note that, by triangle inequality, $a_i < p - a_i$, so

$$\frac{a_i}{p-a_i} < \frac{2a_i}{p}\,,$$

for all $i = 1, , 2 \ldots, n$. Summing this inequality over i we get the desired inequality. □

6. For a natural number n, let $T(n)$ denote the number of ways we can place n objects of weights $1, 2, \ldots, n$ on a balance such that the sum of the weights in each pan is the same. Prove that $T(100) > T(99)$.

Solution. Let $\mathcal{S}(n)$ denote the collection of subsets A of $X(n) = \{1, 2, \ldots, n\}$ such that the sum of the elements of A equals $n(n+1)/4$. Then the given inequality is equivalent to $|\mathcal{S}(100)| > |\mathcal{S}(99)|$. We shall give a map $f : \mathcal{S}(99) \to \mathcal{S}(100)$ which is one-to-one but not onto. Note that this will prove the required inequality.

Suppose that A is an element of $\mathcal{S}(99)$. If $50 \in A$ then define $f(A) = (A \setminus \{50\}) \cup \{100\}$. Otherwise, define $f(A) = A \cup \{50\}$. If A and B are elements of $\mathcal{S}(99)$ such that $f(A) = f(B)$ then either 50 belongs to both these sets or neither of these sets. In either of the cases we have $A = B$. Therefore f is a one-to-one function.

Note that every element in the range of f contains exactly one of 50 and 100. Let $B_i = \{i, 101 - i\}$. Then $B_1 \cup B_2 \cup \cdots B_{24} \cup B_{50}$ is an element of $\mathcal{S}(100)$. Clearly, this is not in the range of f. Thus f is not an onto function. □

——— ⋆⋆ ———