

**NATIONAL BOARD FOR HIGHER MATHEMATICS**  
AND  
**HOMI BHABHA CENTRE FOR SCIENCE EDUCATION**  
**TATA INSTITUTE OF FUNDAMENTAL RESEARCH**

**Pre-REGIONAL MATHEMATICAL OLYMPIAD, 2013**  
**Mumbai Region**

**October 20, 2013**

**QUESTION PAPER SET: B**

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- There are 20 questions in this question paper. Each question carries 5 marks.
  - Answer all questions.
  - Time allotted: 2 hours.
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**QUESTIONS**

1. Let  $S_n = \sum_{k=0}^n \frac{1}{\sqrt{k+1} + \sqrt{k}}$ . What is the value of  $\sum_{n=1}^{99} \frac{1}{S_n + S_{n-1}}$ ?
2. Three points  $X, Y, Z$  are on a straight line such that  $XY = 10$  and  $XZ = 3$ . What is the product of all possible values of  $YZ$ ?
3. It is given that the equation  $x^2 + ax + 20 = 0$  has integer roots. What is the sum of all possible values of  $a$ ?
4. What is the smallest positive integer  $k$  such that  $k(3^3 + 4^3 + 5^3) = a^n$  for some positive integers  $a$  and  $n$ , with  $n > 1$ ?
5. There are  $n - 1$  red balls,  $n$  green balls and  $n + 1$  blue balls in a bag. The number of ways of choosing two balls from the bag that have different colours is 299. What is the value of  $n$ ?
6. Let Akbar and Birbal together have  $n$  marbles, where  $n > 0$ .  
Akbar says to Birbal, "If I give you some marbles then you will have twice as many marbles as I will have." Birbal says to Akbar, "If I give you some marbles then you will have thrice as many marbles as I will have."  
What is the minimum possible value of  $n$  for which the above statements are true?
7. Carol was given three numbers and was asked to add the largest of the three to the product of the other two. Instead, she multiplied the largest with the sum of the other two, but still got the right answer. What is the sum of the three numbers?
8. Let  $S(M)$  denote the sum of the digits of a positive integer  $M$  written in base 10. Let  $N$  be the smallest positive integer such that  $S(N) = 2013$ . What is the value of  $S(5N + 2013)$ ?

9. Let  $AD$  and  $BC$  be the parallel sides of a trapezium  $ABCD$ . Let  $P$  and  $Q$  be the midpoints of the diagonals  $AC$  and  $BD$ . If  $AD = 16$  and  $BC = 20$ , what is the length of  $PQ$ ?
10. In a triangle  $ABC$ , let  $H, I$  and  $O$  be the orthocentre, incentre and circumcentre, respectively. If the points  $B, H, I, C$  lie on a circle, what is the magnitude of  $\angle BOC$  in degrees?
11. Let  $m$  be the smallest odd positive integer for which  $1 + 2 + \dots + m$  is a square of an integer and let  $n$  be the smallest even positive integer for which  $1 + 2 + \dots + n$  is a square of an integer. What is the value of  $m + n$ ?
12. Let  $ABC$  be an equilateral triangle. Let  $P$  and  $S$  be points on  $AB$  and  $AC$ , respectively, and let  $Q$  and  $R$  be points on  $BC$  such that  $PQRS$  is a rectangle. If  $PQ = \sqrt{3}PS$  and the area of  $PQRS$  is  $28\sqrt{3}$ , what is the length of  $PC$ ?
13. To each element of the set  $S = \{1, 2, \dots, 1000\}$  a colour is assigned. Suppose that for any two elements  $a, b$  of  $S$ , if 15 divides  $a + b$  then they are both assigned the same colour. What is the maximum possible number of distinct colours used?
14. Let  $A_1, B_1, C_1, D_1$  be the midpoints of the sides of a convex quadrilateral  $ABCD$  and let  $A_2, B_2, C_2, D_2$  be the midpoints of the sides of the quadrilateral  $A_1B_1C_1D_1$ . If  $A_2B_2C_2D_2$  is a rectangle with sides 4 and 6, then what is the product of the lengths of the diagonals of  $ABCD$ ?
15. Three real numbers  $x, y, z$  are such that  $x^2 + 6y = -17$ ,  $y^2 + 4z = 1$  and  $z^2 + 2x = 2$ . What is the value of  $x^2 + y^2 + z^2$ ?
16. In a triangle  $ABC$  with  $\angle BCA = 90^\circ$ , the perpendicular bisector of  $AB$  intersects segments  $AB$  and  $AC$  at  $X$  and  $Y$ , respectively. If the ratio of the area of quadrilateral  $BXYC$  to the area of triangle  $ABC$  is  $13 : 18$  and  $BC = 12$  then what is the length of  $AC$ ?
17. What is the maximum possible value of  $k$  for which 2013 can be written as a sum of  $k$  consecutive positive integers?
18. Let  $S$  be a circle with centre  $O$ . A chord  $AB$ , not a diameter, divides  $S$  into two regions  $R_1$  and  $R_2$  such that  $O$  belongs to  $R_2$ . Let  $S_1$  be a circle with centre in  $R_1$ , touching  $AB$  at  $X$  and  $S$  internally. Let  $S_2$  be a circle with centre in  $R_2$ , touching  $AB$  at  $Y$ , the circle  $S$  internally and passing through the centre of  $S$ . The point  $X$  lies on the diameter passing through the centre of  $S_2$  and  $\angle YXO = 30^\circ$ . If the radius of  $S_2$  is 100 then what is the radius of  $S_1$ ?
19. Let  $f(x) = x^3 - 3x + b$  and  $g(x) = x^2 + bx - 3$ , where  $b$  is a real number. What is the sum of all possible values of  $b$  for which the equations  $f(x) = 0$  and  $g(x) = 0$  have a common root?
20. What is the sum (in base 10) of all the natural numbers less than 64 which have exactly three ones in their base 2 representation?

The answer key will be uploaded on October 21, 2013 at  
<http://olympiads.hbcse.tifr.res.in/subjects/mathematics/pre-rmo-2013-answers>