Regional Mathematical Olympiad-2016

Time: 3 hours **October 09, 2016**

Instructions:

- **Calculators (in any form) and protractors are not allowed.**
- **Rulers and compasses are allowed.**
- **Answer all the questions.**
- **All questions carry equal marks. Maximum marks: 102.**
- **Answer to each question should start on a new page. Clearly indicate the question number.**

1. Let ABC be a right-angled triangle with $\angle B = 90^\circ$. Let I be the incentre of ABC. Draw a line perpendicular to AI at I. Let it intersect the line CB at D. Prove that CI is perpendicular to AD and prove that $ID = \sqrt{b(b-a)}$ where $BC = a$ and $CA = b$.

2. Let a, b, c be positive real numbers such that
$$\frac{a}{1+a} + \frac{b}{1+b} + \frac{c}{1+c} = 1.$$
Prove that $abc \leq 1/8$.

3. For any natural number n, expressed in base 10, let $S(n)$ denote the sum of all digits of n. Find all natural numbers n such that $n = 2S(n)^2$.

4. Find the number of all 6-digit natural numbers having exactly three odd digits and three even digits.

5. Let ABC be a triangle with centroid G. Let the circumcircle of triangle AGB intersect the line BC in X different from B; and the circumcircle of triangle AGC intersect the line BC in Y different from C. Prove that G is the centroid of triangle AXY.

6. Let $\langle a_1, a_2, a_3, \ldots \rangle$ be a strictly increasing sequence of positive integers in an arithmetic progression. Prove that there is an infinite subsequence of the given sequence whose terms are in a geometric progression.

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