Rao IIT Academy
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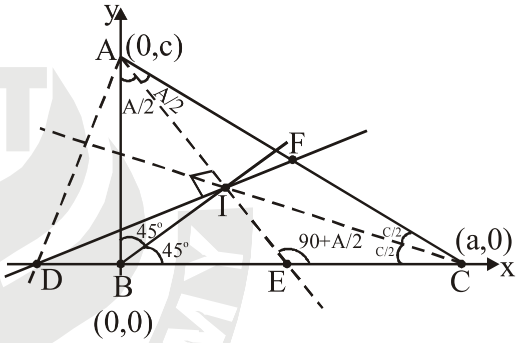
Date: 09/10/2016

RMO TEST

MATHEMATICS

Duration : 3 Hrs. **Maximum Marks : 102**

1. Consider origin at 'B' and axis along BC and BA



In ΔABC,

$\underline{|AEB} = 90 - \frac{A}{2}$

So, $\underline{|AEB} = 180 - \left(90 - \frac{A}{2}\right)$

$= 90 + \frac{A}{2}$

So slope of the line AI is $\tan\left(90 + \frac{A}{2}\right)$

$= -\cot\frac{A}{2}$

as $AB = c$, point 'A' is (0,c)

$BC = a$, point 'C' is (a, 0)

as the line ID is perpendicular to AI,

its slope will be $\tan\frac{A}{2}$ $(\because \text{product of slopes} = 1, \text{ for perpendicular lines})$

<u>equation of line AI :</u>

slope $= -\cot\frac{A}{2}$

it passes through 'A' i.e. (0,c)

So equation is $(y - c) = -\cot\frac{A}{2}.(x - 0)$

$\Rightarrow y = -\cot\frac{A}{2}x + c$ ________(1)

<u>equation of BI :-</u>

slope = tan $45^0 = 1$

passes through (0,0)

So equation is $y = x$_____(2)

Solve the above 2 equations (1), (2) we get the point 'I' as $\left(\frac{c}{1+\cot\frac{A}{2}}, \frac{c}{1+\cot\frac{A}{2}}\right)$

<u>equation of ID :-</u>

we know its slope = $\tan\frac{A}{2}$ (it proved above)

and point 'I' is $\left(\frac{c}{1+\cot\frac{A}{2}}, \frac{c}{1+\cot\frac{A}{2}}\right)$

So equation is $\left(y - \frac{c}{1+\cot\frac{A}{2}}\right) = \tan\frac{A}{2}\left(x - \frac{c}{1+\cot\frac{A}{2}}\right)$

This meets the x-axis at 'D'

So, to find 'D', put $y = 0$

$$\Rightarrow \frac{-c}{1+\cot\frac{A}{2}} = \tan\frac{A}{2}\left(x - \frac{c}{1+\cot\frac{A}{2}}\right)$$

$$\Rightarrow \frac{-c.\cot\frac{A}{2}}{1+\cot\frac{A}{2}} = x - \frac{c}{1+\cot\frac{A}{2}}$$

$$\Rightarrow x = \frac{c}{1+\cot\frac{A}{2}}\left[1-\cot\frac{A}{2}\right] = c\left(\frac{\tan\frac{A}{2}-1}{1+\tan\frac{A}{2}}\right)$$

To prove CI is perpendicular to AD find their slopes.

Slope of AD :

'A' is (0,C)

'D' is $\left(c\left(\frac{1-\cot\frac{A}{2}}{1+\cot\frac{A}{2}}\right),0\right)$

$$\text{Slope} = \frac{-c}{c\left(\frac{1-\cot\frac{\text{A}}{2}}{1+\cot\frac{\text{A}}{2}}\right)}$$

$$= -\left(\frac{1+\cot\frac{A}{2}}{1-\cot\frac{A}{2}}\right)$$

$$= -\left(\frac{\tan\frac{A}{2}+1}{\tan\frac{A}{2}-1}\right)$$

$$= \frac{1+\tan\frac{A}{2}}{1-\tan\frac{A}{2}}$$

$$= \tan\left(45+\frac{A}{2}\right)$$

$$= \tan\left(90-\left(45-\frac{A}{2}\right)\right) \qquad \left[\begin{array}{l} \therefore \text{In } \Delta\text{ABC}, \angle B = 90^0 \\ \text{So}, \angle A + \angle C = 90^0 \\ \text{So}, \frac{A}{2}+\frac{C}{2} = 45^0 \end{array}\right]$$

$$= \tan\left(90-\frac{C}{2}\right)$$

$$= \cot\frac{C}{2}$$

angle made by CI with 'x' axis is $\frac{c}{2}+\left(180-C\right)=180-\frac{c}{2}$

So slope is $\tan\left(180-\frac{c}{2}\right)$

$$=-\tan\frac{c}{2}$$

we can see that product of slopes of CI and AD

$$=\cot\frac{c}{2}\times\left(-\tan\frac{c}{2}\right)=-1$$

So they are perpendicular

length of ID :

Point 'I' is $\left(\frac{c}{1+\cot\frac{A}{2}},\frac{c}{1+\cot\frac{A}{2}}\right)$

'D' is $\left(c\left(\frac{1-\cot\frac{A}{2}}{1+\cot\frac{A}{2}}\right),0\right)$

$$\text{length}=\sqrt{\left(\frac{c}{1+\cot\frac{A}{2}}-\frac{c\left(1-\cot\frac{A}{2}\right)}{1+\cot\frac{A}{2}}\right)^2+\left(\frac{c}{1+\cot\frac{A}{2}}\right)^2}$$

$$=\frac{c}{1+\cot\frac{A}{2}}\sqrt{\left(1-1+\cot\frac{A}{2}\right)^2+1}$$

$$=\frac{c}{1+\cot\frac{A}{2}}\sqrt{1+\cot^2\frac{A}{2}}$$

$$=c\sqrt{\frac{1+\cot^2\frac{A}{2}}{\left(1+\cot\frac{A}{2}\right)^2}}$$

$$= c\sqrt{\frac{1}{1+\frac{2\cot\frac{A}{2}}{1+\cot^2\frac{A}{2}}}}$$

$$= c\sqrt{\frac{1}{1+\frac{2\cot\frac{A}{2}}{1+\tan^2\frac{A}{2}}}}$$

$$= c\sqrt{\frac{1}{1+\sin A}} \qquad \left(\therefore \sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$= \sqrt{c^2\left(\frac{1}{1+\sin A}\right)}$$

In ΔABC $b^2 = a^2 + c^2$ and sinA = $\frac{a}{b}$ because it is right angled triangle.

$$= \sqrt{\left(b^2-a^2\right)\left(\frac{1}{1+\frac{a}{b}}\right)} = \sqrt{(b-a)(b+a)\frac{b}{b+a}} = \sqrt{b(b-a)}$$

2. $a, b, c \in R^+$

$\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c}=1$, P.T. $abc \le \frac{1}{8}$

equation (1) $\rightarrow \frac{a}{1+a}+\frac{b}{1+b}=1-\frac{c}{1+c}$

$$=\frac{1}{1+c}$$

equation (2) $\rightarrow \frac{a}{1+a}+\frac{c}{1+c}=1-\frac{b}{1+b}$

$$=\frac{1}{1+b}$$

equation (3) $\rightarrow \frac{b}{1+b}=\frac{c}{1+c}=1-\frac{a}{1+a}$

$$=\frac{1}{1+a}$$

(1) + (2) + (3)

$$\Rightarrow 2\left[\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c}\right]=\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}$$

$$\Rightarrow \frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}=2 \rightarrow (4)$$

Now consider given equation, $\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c}=1$

Apply A.M, G .M. inequality

A.M $\geq$ G.M

$$\Rightarrow \frac{\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c}}{3} \geq \sqrt[3]{\frac{abc}{(1+a)(1+b)(1+c)}}$$

$$\Rightarrow \frac{1}{3} \geq \sqrt[3]{\frac{abc}{(1+a)(1+b)(1+c)}} \quad \text{\_\_\_\_\_\_\_\_(5)}$$

Now consider equation (4)

$$\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}=2$$

Apply A.M, G.M inequality

A.M. $\geq$ G.M.

$$\Rightarrow \frac{\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}}{3} \geq \sqrt[3]{\frac{1}{(1+a)}\cdot\frac{1}{(1+b)}\cdot\frac{1}{(1+c)}}$$

$$\Rightarrow \frac{2}{3} \geq \sqrt[3]{\frac{1}{(1+a)(1+b)(1+c)}} \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_(6)}$$

divide the equation (5) and (6) i.e. $\frac{(5)}{(6)}$

$$\Rightarrow \frac{\frac{1}{3}}{\frac{2}{3}} \geq \frac{\sqrt[3]{\frac{abc}{(1+a)(1+b)(1+c)}}}{\sqrt[3]{\frac{1}{(1+a)(1+b)(1+c)}}}$$

$\Rightarrow \frac{1}{2} \geq \sqrt[3]{abc}$

$\Rightarrow \sqrt[3]{abc} \leq \frac{1}{2}$

cubing on both sides

$\Rightarrow abc \leq \frac{1}{8}$

Hence proved.

3. Given $n = 2\left(s\left(n\right)\right)^2$

So $\frac{n}{2}$ is a perfect square

So make the table and check

first for 2 digit numbers

So, $\frac{n}{2}$ will be between 5 and 50

If n is greater than 4 digits number.

$\frac{n}{2} > \frac{10000}{2} = 5000$

but $(a+b+c+d+e)$ max value is 2025

So not possible

So n is not greater than 4 digits.

$\frac{n}{2}$	$s(n)$	$2(s)^2$	$n = 2\left(s\left(n\right)\right)^2$
16	5	50	No
25	5	50	yes
36	9	162	No
49	17	$2(17)^2$	No

Same method for three digit numbers.

$\frac{n}{2}$ will be between 50 to 500

$\frac{n}{2}$	$s(n)$	$s(n)^2$	
64	11	121	No
81	9	81	Yes
100	1	1	No
121	4	16	No
144	9	81	
169	16	16^2	No
196	16	16^2	No
225	9	81	No
256	13	169	No
289	19	361	No
324	18	324	Yes
361	10	100	No
400	4	16	No
441	9	81	No
484	16	256	No
529	16	256	No

Only two values of n

$\frac{n}{2}=81$

$\Rightarrow n=162$

$\frac{n}{2}=324$

$\Rightarrow n=648$ in 3 digit numbers.

No. 4 digit numbers are possible

Since suppose $n=abcd$

$\frac{n}{2}=(a+b+c+d)^2$

Since 4 digit number $a \geq 1$

but $(a+b+c+d)$ max value $=(36)^2=1296$

$(a=b=c=d=9)$

So $n <$ (1296) x 2

So $a = 1$, 2 only

now take $a = 2$

$(2+b+c+d)^2 \Rightarrow$ max value is $(29)^2$ $(\because b=c=d=a)$

$\Rightarrow \frac{n}{2} \le (29)^2 \Rightarrow n < 2000$ not possible

So $a = 1$ only

Now since $a = 1$

$\frac{n}{2} = (1+b+c+d)^2$

$(28)^2 \to$ max value of $(s(n))^2$

$\Rightarrow n$ can have max value of 1568

$n = a+b+c+d > 23$ Since $n > 100$

$\Rightarrow 1+b+c+d > 22+1$

$\Rightarrow b+c+d > 22$

So $b = 4,5$ only

$b \ge 22-18 \quad b \ge 4$

the squares in 700 to 800 donot satisfy so no possible 4 digit solutions to our answer.

4. 6-digit natural number ---------------

from 6 digit, 3 digits should be even, sol selection of 3 digits is $^6C_3$, $^3C_3$.

Now, there are 5 even and 5 odd digits

Selection of 3 even digits (can be repeatable) $= 5\times5\times5$

Selection of 3 odd digits from $5 = 5\times5\times5$

Total number of 6-digit number $= 6C_3.3C_3.5C^3 5^3$

$= 20\times125\times125$

$= 3,12,500$

Now, from these number we have to eliminate those numbers starting with zero.

Case I : fix first digit as '0'. There will be '5' digits left.

Selecting '2' even digits in 5 empty places   O_ _ _ _ _ _ _ _

$= {}^5C_2 . {}^3C_3$ (selecting of odd digit places $^3C_3$)

Selection of 2 even digits from 5 numbers $= 5\times5$

(since '0' is selected already)

Selection of 3 odd digits from 5 numbers $= 5\times5\times5$

Total number of 6-digits numbers

starting with zero $= {}^5C_2.{}^3C_3.5^2\times5^3$

$= 31250$

Case II : fix first two digits as '0' 4 digits left.

Selecting one even digit from 4 digits = ${}^4C_1$ 0 0 _ _ _ _ _ _

Selecting three odd digits from 3 digits = ${}^3C_3$

Selection of 1 even digit from 5 number = 5

Selection of 3 odd digits from 5 number = $5\times5\times5$

Total number of 6 digit numbers

Starting with two zeroes = ${}^4C_1.{}^3C_3.5\times5^3$

$= 2500$

Case III : fix first three digit as '0'. 3 digit left

Since we have taken 3 even digits as '0'. So selection of 3 left digits will be odd and is ${}^3C_3$

Selecting of 3 odd digits from 5 numbers $= 5\times5\times5$ 0 0 0 _ _ _ _ _

Total number of 6 digit numbers

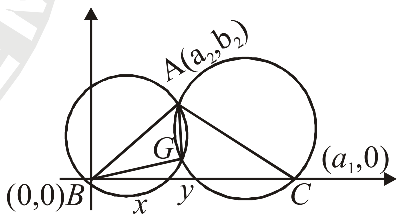
Starting with three zeros = ${}^3C_3.125 = 125$

Now, finally

Required number of all 6-digit natural number having exactly three even digits three odd digits will be.

$312500 - 31250 + 2500 - 125 = 283625$

5. Let define a triangle ABC with vertices $B(0,0), C(a_1,0), A(a_2,b_2)$



Centroid of this $\Delta = \left(\dfrac{a_1+a_2}{3}, \dfrac{b_2}{3}\right) = G$

Now, Circumcircle of $\Delta\ AGB$ intersects line BC in x different from B.

Similarly circumcircle of ΔACG intersects line BC in y different from C.

let $X = (x_1, 0), \quad Y = (y_1, 0)$

Now centroid of $\Delta AXY = \left(\dfrac{x_1+a_2}{3}, \dfrac{b_2}{3}\right) = G_1$

Now centroid of $\Delta AYC = \left(\dfrac{x_1+y_1+a_2}{3}, \dfrac{b_2}{3}\right) = G_2$

Now centroid of $\Delta AYC = \left(\dfrac{y_1+a_1+a_2}{3}, \dfrac{b_2}{3}\right) = G_3$

Now, triangle formed by these three centroids will be having same centroid as of triangle ABC. Since triangle ABC splits 3 triangles centroid of triangle formed by G_1, G_2, G_3

$$\left[\frac{\frac{x_1+a_2}{3}+\frac{a_2+x_1+y_1}{3}+\frac{y_1+a_1+a_2}{3}}{3}, \frac{\frac{b_2}{3}+\frac{b_2}{3}, \frac{b_2}{3}}{3}\right]=\left[\frac{a_1+a_2}{3}, \frac{b_2}{3}\right]$$

$$\left[\frac{2(x_1+y_1)+3a_2+a_1}{9}, \frac{b_2}{3}\right]=\left[\frac{a_1+a_2}{3}, \frac{b_2}{3}\right]$$

$$\frac{2(x_1+y_1)}{9}+\frac{a_2}{3}+\frac{a_1}{9}=\frac{a_1}{3}+\frac{a_2}{3}$$

$$\frac{2(x_1+y_1)}{9}=\frac{3a_1-a_1}{9}$$

$$(x_1+y_1)=a_1$$

Centroid of $\Delta AXY=\left[\frac{x_1+y_1+a_2}{{}^{3}3}, \frac{b^2}{3}\right]$

$$=\left(\frac{a_1+a_2}{3}, \frac{b_2}{3}\right)$$

$$=G$$

$=$ Centroid of ΔABC

6. $\{a_1, a_2, a_3......\}$ are terms of an AP such that they are all + ve integers.

Let AP series is

$a, a+d, a+2d.......a+nd$

As $\frac{a}{d}$ is rational number

let $\frac{a}{d}=\frac{p}{q}$ $(p\neq 0)$

$$a+nd=a+n\times a\times\left(\frac{q}{p}\right)=a\left\{1+n\left(\frac{q}{p}\right)\right\}$$

GP will be of the form ar, ar^2, ar^3

let $1+n\left(\frac{q}{p}\right)=r$

when n is multiple of p then r will be integer

Now, $r^2=1+n^2\left(\frac{q}{p}\right)^2+2n\left(\frac{q}{p}\right)$

Also, if n is multiple of p then

r^2 will also be an integer

So, r^2 can be written as

$$r^2 = 1 + m\left(\frac{q}{p}\right); \quad m \in I$$

$$\text{then } ar^2 = \underbrace{a\left[1 + m\left(\frac{q}{p}\right)\right]}_{\text{this term will be from AP.}}$$

Again :

$$r^3 = 1 + n^3\left(\frac{q}{p}\right)^3 + 3n\left(\frac{q}{p}\right)\left[n\left(\frac{q}{p}\right) + 1\right]$$

r^3 will be an integer when n will be multiple of p.

So, $r^3 = 1 + s\left(\frac{q}{p}\right) \qquad 5 \in I$

$\therefore ar^3 = a\left[1 + s\left(\frac{q}{p}\right)\right]$ which is from AP. Therefore ar, ar^2, ar^3 are general units of GP which is from given AP.

There are infinite number of r which will satisfy above condition.